Problem Set 8

Problem 1: Uniform plane wave [+30 pts]

Consider a uniform plane wave propagating in free space with the following electric field:

$$\overrightarrow{E}(y,z,t) = Re\left\{\hat{x}E_0 e^{j\omega t} e^{-\frac{jk(y+z)}{\sqrt{2}}}\right\}$$

where $k = \frac{\omega}{c}$.

a **Obtain the direction of propagation and its unit vector $\hat{k}$.**

We can write the phasor, $\tilde{E}$, as:

$$\tilde{E} = \hat{x}E_0 e^{-\frac{jk(y+z)}{\sqrt{2}}}$$

Where we know for a uniform plane wave, the general form of the exponential is

$$e^{-j\overrightarrow{k}\cdot\overrightarrow{r}}$$

And can write

$$\overrightarrow{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

Thus we can solve for $\overrightarrow{k}$

$$\overrightarrow{k}\cdot\overrightarrow{r} = \frac{k(y+z)}{\sqrt{2}} \Longrightarrow \overrightarrow{k} = \frac{k}{\sqrt{2}}(\hat{y}+\hat{z})$$

$\boxed{\text{The direction of } \overrightarrow{k}\text{, and thus of propagation, is } \hat{y}+\hat{z}.}$

We can get the uniform plane wave's unit vector, $\hat{k}$, by:

$$\hat{k} = \frac{\overrightarrow{k}}{|\overrightarrow{k}|} = \frac{\frac{k}{\sqrt{2}}(\hat{y}+\hat{z})}{k} = \frac{1}{\sqrt{2}}(\hat{y}+\hat{z})$$

b **Find the magnetic field $H(y,z,t)$ and its phasor.**

First, we can solve for the magnetic field phasor:

$$\begin{aligned}\tilde{H} &= \frac{1}{\eta}\hat{k}\times\tilde{E} \\ &= \frac{1}{\eta}\frac{1}{\sqrt{2}}(\hat{y}+\hat{z})\times\hat{x}E_0 e^{-j\frac{k}{\sqrt{2}}(y+z)} \\ &= \frac{E_0}{\eta\sqrt{2}}e^{-j\frac{k}{\sqrt{2}}(y+z)}(\hat{y}-\hat{z})\end{aligned}$$

We are in free space so $\eta = \eta_0 \approx 377\Omega$.

To solve for the magnetic field vector (which shares the same ω).

$$\begin{aligned}
\overrightarrow{H} &= Re\left\{\tilde{H}e^{j\omega t}\right\} \\
&= Re\left\{\frac{E_0}{\eta\sqrt{2}}e^{-j\frac{k}{\sqrt{2}}(y+z)}(\hat{y}-\hat{z})e^{j\omega t}\right\} \\
&= Re\left\{\frac{E_0}{\eta\sqrt{2}}e^{j\left(\omega t-\frac{k}{\sqrt{2}}(y+z)\right)}(\hat{y}-\hat{z})\right\} \\
&= \frac{|E_0|}{\eta\sqrt{2}}\cos\left(\omega t-\frac{k}{\sqrt{2}}(y+z)+\phi_0\right)(\hat{y}-\hat{z}) \\
&= \frac{|E_0|}{\eta_0\sqrt{2}}\cos\left(\omega t-\frac{k}{\sqrt{2}}(y+z)\right)(\hat{y}-\hat{z})
\end{aligned}$$

Because η is fully real, the phase offset inside the cosine term due to the intrinsic impedance is 0. For complex intrinsic impedances, this term will not be 0.

c **Specify the polarization state of the electric field.**
The polarization state of the electric field is the direction of the electric field, which is linearly polarized in the $\hat{x}$ direction.

Problem 2: Electromagnetic Wave Polarization [+40 pts]

The electric field complex phasor of a uniform plane wave propagating in free space is given by:

$$\tilde{E} = (\hat{x}+j\hat{y})30e^{-j\left(\frac{\pi}{6}z+\frac{\pi}{2}\right)} \qquad \left[\frac{V}{m}\right]$$

a **Obtain the complete expression for the electric field, $\overrightarrow{E}(z,t)$, including solving for the angular frequency, ω in terms of known values and physical constants.**
The most straightforward way to compute this is to expand the exponential via Euler's formula and multiply through until we can completely separate the imaginary and real components.

$$\begin{aligned}
\overrightarrow{E}(z,t) &= \Re\left\{\tilde{E}e^{j\omega t}\right\} \\
&= \Re\left\{(\hat{x}+j\hat{y})30e^{-j\left(\frac{\pi}{6}z+\frac{\pi}{2}\right)}e^{j\omega t}\right\} \\
&= \Re\left\{(\hat{x}+j\hat{y})30e^{j\left(\omega t-\frac{\pi}{6}z-\frac{\pi}{2}\right)}\right\} \\
&= 30\Re\left\{e^{j\left(\omega t-\frac{\pi}{6}z-\frac{\pi}{2}\right)}\hat{x}+je^{j\left(\omega t-\frac{\pi}{6}z-\frac{\pi}{2}\right)}\hat{y}\right\} \\
&= 30\Re\left\{e^{j\left(\omega t-\frac{\pi}{6}z-\frac{\pi}{2}\right)}\hat{x}+e^{j\frac{\pi}{2}}e^{j\left(\omega t-\frac{\pi}{6}z-\frac{\pi}{2}\right)}\hat{y}\right\} \\
&= 30\Re\left\{e^{j\left(\omega t-\frac{\pi}{6}z-\frac{\pi}{2}\right)}\hat{x}+e^{j\left(\omega t-\frac{\pi}{6}z\right)}\hat{y}\right\} \\
\overrightarrow{E}(z,t) &= 30\left[\cos\left(\omega t-\frac{\pi}{6}z-\frac{\pi}{2}\right)\hat{x}+\cos\left(\omega t-\frac{\pi}{6}z\right)\hat{y}\right]
\end{aligned}$$

We know this is an electromagnetic field propagating in free space, so we can use the relation $k = \frac{\omega}{c}$ where c is the speed of light in free space.

$$\begin{aligned} k &= \frac{\pi}{6} \\ \omega &= ck = 3 \times 10^8 \frac{\pi}{6} \\ &= \frac{\pi}{2} \times 10^8 \end{aligned}$$

So plugging that into our equation above.

$$\overrightarrow{E}(z,t) = 30\left[\cos\left(\frac{\pi}{2} \times 10^8 t - \frac{\pi}{6}z - \frac{\pi}{2}\right)\hat{x} + \cos\left(\frac{\pi}{2} \times 10^8 t - \frac{\pi}{6}z\right)\hat{y}\right]$$

b **Obtain the corresponding expression for the magnetic field,** $\overrightarrow{H}(z,t)$**.**

With $\hat{k} = \hat{z}$, we can solve for the magnetic field phasor:

$$\begin{aligned} \tilde{H} &= \frac{1}{\eta}\hat{k} \times \tilde{E} \\ &= \frac{1}{\eta}(\hat{z}) \times (\hat{x} + j\hat{y}) 30 e^{-j\left(\frac{\pi}{6}z + \frac{\pi}{2}\right)} \\ &= \frac{30}{\eta} e^{-j\left(\frac{\pi}{6}z + \frac{\pi}{2}\right)}(\hat{y} - j\hat{x}) \end{aligned}$$

Converting to the time domain, remembering that we are in free space, so $\eta = \eta_0 \approx 377\Omega$:

$$\begin{aligned} \overrightarrow{H} &= \Re\left\{\tilde{H} e^{j\omega t}\right\} \\ &= \Re\left\{\frac{30}{\eta} e^{-j\left(\frac{\pi}{6}z + \frac{\pi}{2}\right)}(\hat{y} - j\hat{x}) e^{j\omega t}\right\} \\ &= \frac{30}{\eta}\Re\left\{e^{j\left(\omega t - \frac{\pi}{6}z - \frac{\pi}{2}\right)}\hat{y} - j e^{j\left(\omega t - \frac{\pi}{6}z - \frac{\pi}{2}\right)}\hat{x}\right\} \\ &= \frac{30}{\eta}\Re\left\{e^{j\left(\omega t - \frac{\pi}{6}z - \frac{\pi}{2}\right)}\hat{y} - e^{j\left(\omega t - \frac{\pi}{6}z\right)}\hat{x}\right\} \\ &= \frac{30}{\eta}\left[\cos\left(\omega t - \frac{\pi}{6}z - \frac{\pi}{2}\right)\hat{y} - \cos\left(\omega t - \frac{\pi}{6}z\right)\hat{x}\right] \\ &= \frac{30}{\eta}\left[\cos\left(\frac{\pi}{2} \times 10^8 t - \frac{\pi}{6}z - \frac{\pi}{2}\right)\hat{y} - \cos\left(\frac{\pi}{2} \times 10^8 t - \frac{\pi}{6}z\right)\hat{x}\right] \end{aligned}$$

c **Specify the magnitude and direction of the electric field at the** $z = 0$ **plane for times,** $t = 0, 5, 10 \quad [ns]$ **(**$ns = 10^{-9}s$**).**

The magnitude is always equal to 30, the direction rotates in time.

Next we plug in $z = 0$ and $t = 0, 5, 10$ ns:

For $z = 0, t = 0$:

$$\overrightarrow{E}(0,0) = 30\left[\cos\left(\frac{\pi}{2} \times 10^8(0) - \frac{\pi}{6}(0) - \frac{\pi}{2}\right)\hat{x} + \cos\left(\frac{\pi}{2} \times 10^8(0) - \frac{\pi}{6}(0)\right)\hat{y}\right] = 30\hat{y}$$

For $z = 0, t = 5ns$:

$$\overrightarrow{E}(0, 5ns) = 30\left[\cos\left(\frac{\pi}{2} \times 10^8(5 \times 10^{-9}) - \frac{\pi}{6}(0) - \frac{\pi}{2}\right)\hat{x} + \cos\left(\frac{\pi}{2} \times 10^8(5 \times 10^{-9}) - \frac{\pi}{6}(0)\right)\hat{y}\right]$$
$$= 30(\frac{\hat{x}}{\sqrt{2}} + \frac{\hat{y}}{\sqrt{2}})$$

For $z = 0, t = 10ns$:

$$\overrightarrow{E}(0, 10ns) = 30\left[\cos\left(\frac{\pi}{2} \times 10^8(10 \times 10^{-9}) - \frac{\pi}{6}(0) - \frac{\pi}{2}\right)\hat{x} + \cos\left(\frac{\pi}{2} \times 10^8(10 \times 10^{-9}) - \frac{\pi}{6}(0)\right)\hat{y}\right]$$
$$= 30\hat{x}$$

The direction of the electric field is $\hat{y}$ at $z = 0, t = 0$.
The direction of the electric field is $\hat{x} + \hat{y}$ at $z = 0, t = 5ns$.
The direction of the electric field is $\hat{x}$ at $z = 0, t = 10ns$.

d **Obtain the polarization state of the electric field.**

The polarization state is circularly polarized as the orthogonal components in the $\hat{x}$ and $\hat{y}$ directions are equal in magnitude and $\frac{\pi}{2}$ out of phase relative to one another. We can tell from part b) that the polarization state is left hand circularly polarized, as the polarization state rotates clockwise if you orient the z axis as out of towards yourself (out of the page).

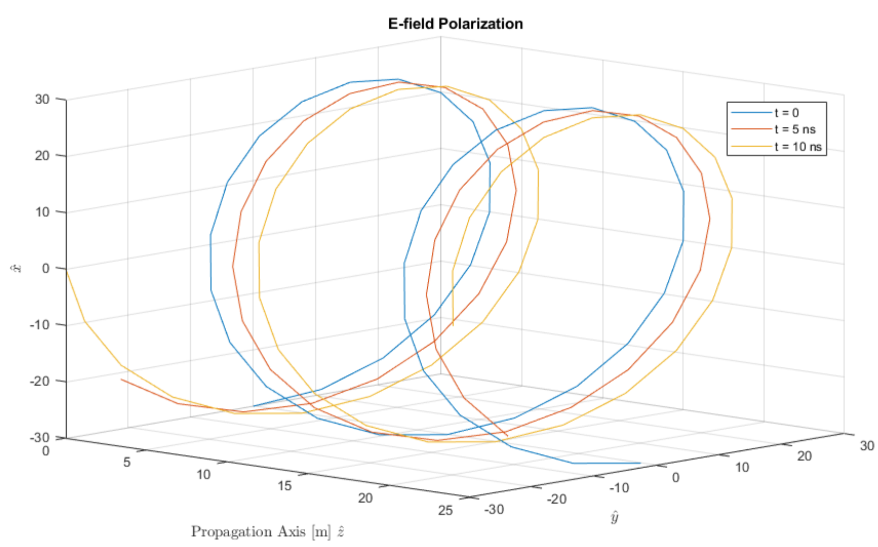

Figure 1: Circularly polarized light propagating through space in the $\hat{z}$ direction at several snap shots in time.

Problem 3: Plane wave propagation in a lossy medium [+30 pts]

The magnetic field of a linearly polarized uniform plane wave propagating in the $+\hat{y}$ direction in sea water, is given (at $y = 0$) by:

$$\overrightarrow{H} = \hat{x}0.1\sin\left(10^{10}\pi t - \frac{\pi}{3}\right) \quad \left[\frac{A}{m}\right] \tag{1}$$

a **The sea water is specified by the following parameters: $\epsilon_r = 80$, $\mu_r = 1$, $\sigma = 4\frac{S}{m}$.**

b **Determine whether this is a lossless medium, a low-loss medium, or a good conductor.**

We can see from the equation for $\overrightarrow{H}$ that $\omega = 10^{10}\pi$, to solve for α and β we need the ratio $\frac{\epsilon"}{\epsilon'}$

$$\frac{\epsilon"}{\epsilon'} = \frac{\sigma}{\omega\epsilon} = \frac{\sigma}{\omega\epsilon_r\epsilon_0} = \frac{4}{10^{10}\pi \cdot 80 \cdot 8.85 \cdot 10^{-12}} = 0.18 \tag{2}$$

which, given the condition $0.01 \leq \frac{\epsilon"}{\epsilon'} \leq 100$, the medium is a quasi conductor.

c **Determine the attenuation constant, α, the phase constant, β, and the complex intrinsic impedance, η_c.**

We can solve for α, β, and η_c with the following equations

$$\alpha = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1 + \left(\frac{\epsilon"}{\epsilon'}\right)^2} - 1\right]\right\}^{\frac{1}{2}} \tag{3}$$

$$\beta = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1 + \left(\frac{\epsilon"}{\epsilon'}\right)^2} + 1\right]\right\}^{\frac{1}{2}} \tag{4}$$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon'}}\left(1 - j\frac{\epsilon"}{\epsilon'}\right)^{-\frac{1}{2}} \tag{5}$$

We have all the required numbers so we can simply plug in $\omega = 10^{10}\pi$, $\mu = \mu_r\mu_0 = \mu_0$, $\epsilon' = \epsilon_r\epsilon_0 = 80\epsilon_0$, and $\frac{\epsilon"}{\epsilon'} = 0.18$.

$$\begin{aligned} \alpha &= 84.02 \quad [\frac{Np}{m}] \\ \beta &= 941.05 \quad [\frac{rad}{m}] \\ \eta_c &= 41.79 \cdot e^{j0.09} \quad [\Omega] \end{aligned} \tag{6}$$

d **Obtain the wavelength λ, and phase velocity, u_p.**

$$u_p = \frac{\omega}{\beta} = 3.34 \times 10^7 \quad [\frac{m}{s}] \tag{7}$$

And

$$\lambda = \frac{u_p}{f} = \frac{2\pi u_p}{\omega} = 6.68 \times 10^{-3} \quad [m] \tag{8}$$

e **Obtain the skin depth, δ_s, and explain why we typically do not use radio frequency or optical communication underseas (e.g. between submarines).**

We know that:

$$\delta_s = \frac{1}{\alpha} = 0.012 \quad [m] \tag{9}$$

This shows us that if we used high frequencies, such as RF or optics, to communicate underseas we would face extremely high attenuation. The amplitude of the communications signal would go to 0 long before it could be received across any meaningful distance.

f **Obtain an expression for $\overrightarrow{E}(y, t)$.**

First let's determine the direction of $\overrightarrow{E}(0, t)$

$$\tilde{H} = \frac{1}{\eta}\hat{k} \times \tilde{E} \tag{10}$$

$\tilde{H}$ is in the $\hat{x}$ direction, and $\hat{k} = \hat{y}$ (direction of propagation), so we know that the direction of $\tilde{E}$ is $\pm\hat{z}$. To determine the sign, we can try to plug one in to get the right direction for $\tilde{H}$.

$$\hat{y} \times \hat{z} = \hat{x} \tag{11}$$

Therefore, the direction of $\tilde{E}$ is $\hat{z}$.

In a lossy medium we can write the phasor form of $\overrightarrow{E}$ and $\overrightarrow{H}$

$$\tilde{E}(z) = \hat{z} \cdot E_{z0} e^{-\alpha y} e^{-j\beta y} e^{j\phi} \tag{12}$$

$$\tilde{H}(z) = \hat{x} \cdot \frac{E_{z0}}{\eta_c} e^{-\alpha y} e^{-j\beta y} e^{j\phi} \tag{13}$$

We know that at $y = 0$

$$\frac{E_{z0}}{\eta_c} = 0.1 e^{-j(\frac{\pi}{3})} e^{-j\frac{\pi}{2}} \tag{14}$$

Plugging in our known value for η_c

$$\begin{aligned} E_{z0} &= 0.1 e^{-j\frac{\pi}{3}} \cdot 41.79 e^{j0.09} e^{-j\frac{\pi}{2}} \\ &= 4.179 e^{-j2.53} \end{aligned} \tag{15}$$

Now we can plug this value into our equation for the phasor $\tilde{E}$ and then transform it back to $\overrightarrow{E}(y,t)$

$$\begin{aligned}\overrightarrow{E}(y,t) &= \Re\left\{\hat{z}E_{z0}e^{-\alpha y}e^{-j\beta y}e^{j\omega t}\right\} \\ &= \Re\left\{4.179 \cdot e^{-j2.53} \cdot e^{-84.02y} \cdot e^{-j941.05y} \cdot e^{j10^{10}\pi t}\right\} \\ &= \hat{z} \cdot 4.179 e^{-84.02y} \cos\left(10^{10}\pi t - 941.05y - 2.53\right)\end{aligned} \tag{16}$$